

Bounds on the tight-binding approximation for the Gross–Pitaevskii equation with a periodic potential

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Abstract

We justify the validity of the discrete nonlinear Schrödinger equation for the tight-binding approximation in the context of the Gross–Pitaevskii equation with a periodic potential. Our construction of the periodic potential and the associated Wannier functions is based on the previous work [7], while our analysis involving energy estimates and Gronwall’s inequality addresses time-dependent localized solutions on large but finite time intervals.

1 Introduction

We consider the Gross–Pitaevskii (GP) equation with a periodic potential in the form

$$i\phi_t = -\phi_{xx} + V(x)\phi + \sigma|\phi|^2\phi, \quad (1.1)$$

where the solution $\phi : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{C}$ decays to zero sufficiently fast as $|x| \rightarrow \infty$, the potential $V : \mathbb{R} \mapsto \mathbb{R}$ is a bounded 2π -periodic function, and the parameter $\sigma = \pm 1$ is normalized for the cubic nonlinear term. In particular, we consider the piecewise-constant potential in the form

$$V(x) = \begin{cases} \varepsilon^{-2}, & x \in (0, a) \bmod(2\pi) \\ 0, & x \in (a, 2\pi) \bmod(2\pi) \end{cases} \quad (1.2)$$

for some fixed $0 < a < 2\pi$ and small $\varepsilon > 0$. The asymptotic limit of small ε represents the so-called tight-binding approximation, for which the potential $V(x)$ is a periodic sequence of large walls of a non-zero width and the lowest bands in the spectrum of the linear operator $L = -\partial_x^2 + V(x)$ are exponentially narrow with respect to ε . According to the tight-binding approximation [1], time-dependent solutions of the GP equation (1.1) are approximated by the time-dependent solutions of the discrete nonlinear Schrödinger (DNLS) equation in the form

$$i\dot{\phi}_n = \alpha(\phi_{n+1} + \phi_{n-1}) + \sigma\beta|\phi_n|^2\phi_n, \quad (1.3)$$

where α and β are ε -independent constants and the sequence $\{\phi_n(t)\}_{n \in \mathbb{Z}}$ represents a small-amplitude solution $\phi(x, t)$ evaluated at the periodic sequence of potential wells.

We proved in the previous work [7] that stationary localized solutions of the GP equation in the form $\phi(x, t) = \Phi(x)e^{-i\omega t}$ with $\Phi \in H^1(\mathbb{R})$ and $\omega \notin \sigma(L)$ are approximated for small values of ε by stationary

localized solutions of the DNLS equation in the form $\phi_n(t) = \Phi_n e^{-i\Omega t}$, where $\vec{\Phi} \in l^1(\mathbb{Z})$ and Ω is related to the rescaled parameter ω . Here and henceforth, we use the standard notations for the Sobolev space $H^1(\mathbb{R})$ of scalar complex-valued functions equipped with the squared norm

$$\|\phi\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} (|\phi'(x)|^2 + |\phi(x)|^2) dx$$

and the space $l^1(\mathbb{Z})$ of vectors representing complex-valued sequences equipped with the norm $\|\vec{\phi}\|_{l^1(\mathbb{Z})} = \sum_{n \in \mathbb{Z}} |\phi_n|$. In this work, we extend our analysis to time-dependent localized solutions of these equations and prove that the formal tight-binding approximation of [1] can be justified for small values of ε on large but finite time intervals. It will be clear from our analysis that the appropriate space for the time-dependent localized solution of the GP equation (1.1) is associated with the quadratic form generated by operator $-\partial_x^2 + V(x) + 1$. We denote this space by $\mathcal{H}^1(\mathbb{R})$ and equip it with the squared norm

$$\|\phi\|_{\mathcal{H}^1(\mathbb{R})}^2 = \int_{\mathbb{R}} (|\phi'(x)|^2 + V(x)|\phi(x)|^2 + |\phi(x)|^2) dx. \quad (1.4)$$

Since $V(x) \geq 0$ for all $x \in \mathbb{R}$, it is clear that $\|\phi\|_{H^1(\mathbb{R})} \leq \|\phi\|_{\mathcal{H}^1(\mathbb{R})}$.

Our analysis is closely related to the recent works on justifications of nonlinear evolution equations for pulses that exist in space-periodic potentials near edges of spectral bands [3] and in narrow band gaps of one-dimensional [10] and two-dimensional [4] potentials. A similar work in the context of a nonlinear heat equation with a periodic diffusive term was developed in [9] with the invariant manifold reductions. Although the justification of lattice equations for the time-dependent solutions of dissipative (reaction–diffusion) equations can be extended globally for $t \geq 0$, the justification of the DNLS equation can only be carried out for finite time intervals because the GP equation is a conservative (Hamiltonian) system. Reductions to the DNLS equation on a finite lattice for a finite time interval were also discussed in the context of the GP equation with a N -well trapping potential [2].

Methods of our analysis follow closely to arguments from [5] and rely on the Wannier function decomposition from [7] as well as on energy estimates and Gronwall's inequality. The Wannier function decomposition is reviewed in Section 2. The energy estimates and the bounds on the remainder terms are studied in Section 3. The main theorem is formulated in Section 2 and proved in Section 3.

2 Wannier function decomposition

Let $u_l(x; k)$ be a Bloch function of the operator $L = -\partial_x^2 + V(x)$ for the eigenvalue $\omega_l(k)$, such that $l \in \mathbb{N}$, $k \in \mathbb{T} = [-\frac{1}{2}, \frac{1}{2}] \bmod(1)$, $u_l(x + 2\pi; k) = u_l(x; k)e^{i2\pi k}$ for all $x \in \mathbb{R}$, and the following orthogonality and normalization conditions are met

$$\int_{\mathbb{R}} \bar{u}_{l'}(x, k') u_l(x, k) dx = \delta_{l, l'} \delta(k - k'), \quad \forall l, l' \in \mathbb{N}, \quad \forall k, k' \in \mathbb{T}, \quad (2.1)$$

where $\delta_{l, l'}$ is the Kronecker symbol and $\delta(k)$ is the Dirac delta function in the sense of distributions. To normalize uniquely the phase factors of the Bloch functions [6], we assume that $u_l(x; -k) = \bar{u}_l(x; k)$ is chosen as a Bloch function for $\omega_l(-k) = \bar{\omega}_l(k) = \omega_l(k)$.

Since the band function $\omega_l(k)$ and the Bloch function $u_l(x; k)$ are periodic with respect to $k \in \mathbb{T}$ for any $l \in \mathbb{N}$, we represent them by the Fourier series

$$\omega_l(k) = \sum_{n \in \mathbb{Z}} \hat{\omega}_{l, n} e^{i2\pi n k}, \quad u_l(x; k) = \sum_{n \in \mathbb{Z}} \hat{u}_{l, n}(x) e^{i2\pi n k}, \quad \forall l \in \mathbb{N}, \quad \forall k \in \mathbb{T}, \quad (2.2)$$

where the coefficients satisfy the constraints

$$\hat{\omega}_{l,n} = \hat{\omega}_{l,-n} = \hat{\omega}_{l,-n}, \quad \hat{u}_{l,n}(x) = \hat{u}_{l,n}(x), \quad \forall n \in \mathbb{Z}, \quad \forall l \in \mathbb{N}, \quad \forall x \in \mathbb{R} \quad (2.3)$$

and

$$\hat{u}_{l,n}(x) = \hat{u}_{l,n-1}(x - 2\pi) = \hat{u}_{l,0}(x - 2\pi n), \quad \forall n \in \mathbb{Z}, \quad \forall l \in \mathbb{N}, \quad \forall x \in \mathbb{R}. \quad (2.4)$$

The real-valued functions $\hat{u}_{l,n}(x)$ are referred to as the Wannier functions. The following two propositions from [7] summarize properties of the band and Wannier functions for the potential $V(x)$ given by (1.2) in the limit of small $\varepsilon > 0$.

Proposition 1 *Let V be given by (1.2). For any fixed $l \in \mathbb{N}$, there exist $\varepsilon_0 > 0$ and ε -independent constants $\zeta_0, \omega_0, c_n > 0$, such that, for any $\varepsilon \in [0, \varepsilon_0)$, the band functions of the operator $L = -\partial_x^2 + V(x)$ satisfy the properties:*

$$(i) \quad (\text{band separation}) \quad \min_{\forall m \in \mathbb{N} \setminus \{l\}} \inf_{\forall k \in \mathbb{T}} |\omega_m(k) - \hat{\omega}_{l,0}| \geq \zeta_0, \quad (2.5)$$

$$(ii) \quad (\text{band boundness}) \quad |\hat{\omega}_{l,0}| \leq \omega_0, \quad (2.6)$$

$$(iii) \quad (\text{tight-binding approximation}) \quad |\hat{\omega}_{l,n}| \leq c_n \varepsilon^n e^{-\frac{na}{\varepsilon}}, \quad n \in \mathbb{N}. \quad (2.7)$$

Proposition 2 *Let V be given by (1.2). For any fixed $l \in \mathbb{N}$, there exists $\varepsilon_0 > 0$ and ε -independent constants $U_0, C_0, C_n > 0$, such that, for any $\varepsilon \in [0, \varepsilon_0)$, the Wannier functions of the operator $L = -\partial_x^2 + V(x)$ satisfy the properties:*

$$(i) \quad (\text{boundness of norms}) \quad \|\hat{u}_{l,n}\|_{\mathcal{H}^1(\mathbb{R})} \leq U_0, \quad n \in \mathbb{N}, \quad (2.8)$$

$$(ii) \quad (\text{compact support}) \quad |\hat{u}_{l,0}(x) - \hat{u}_0(x)| \leq C_0 \varepsilon, \quad \forall x \in [0, 2\pi], \quad (2.9)$$

$$(iii) \quad (\text{exponential decay}) \quad |\hat{u}_{l,0}(x)| \leq C_n \varepsilon^n e^{-\frac{na}{\varepsilon}}, \quad (2.10)$$

$$\forall x \in [-2\pi n, -2\pi(n-1)] \cup [2\pi n, 2\pi(n+1)], \quad n \in \mathbb{N},$$

where

$$\hat{u}_0(x) = \begin{cases} 0, & \forall x \in [0, a], \\ \frac{\sqrt{2}}{\sqrt{2\pi-a}} \sin \frac{\pi l(2\pi-x)}{2\pi-a}, & \forall x \in [a, 2\pi]. \end{cases} \quad (2.11)$$

Figure 1 illustrates the spectrum of L and the Wannier functions for the potential V in (1.2) with $a = \pi$ and $\varepsilon = 0.5$. The left panel shows the first spectral bands of L computed from the trace of the monodromy matrix [7]. The right panel shows the Wannier function $\hat{u}_{1,0}(x)$ computed by using the integral representation $\hat{u}_{1,0}(x) = \int_{\mathbb{T}} u_1(x; k) dk$ and the finite-difference approximation of the Bloch function $u_1(x; k)$. The solid lines for the Wannier function $\hat{u}_{1,0}(x)$ approach the dotted line for the asymptotic approximation (2.11) with $l = 1$ as ε gets smaller.

We use the set of Wannier functions $\{\hat{u}_{l,n}\}_{n \in \mathbb{Z}}$ for a fixed $l \in \mathbb{N}$ to represent formally solutions of the GP equation (1.1) in the form

$$\phi(x, t) = \mu^{1/2} (\varphi_0(x, T) + \mu \varphi(x, t)) e^{-i\hat{\omega}_{l,0}t}, \quad T = \mu t, \quad \mu = \varepsilon e^{-\frac{a}{\varepsilon}}, \quad (2.12)$$

where

$$\varphi_0(x, T) = \sum_{n \in \mathbb{Z}} \phi_n(T) \hat{u}_{l,n}(x). \quad (2.13)$$

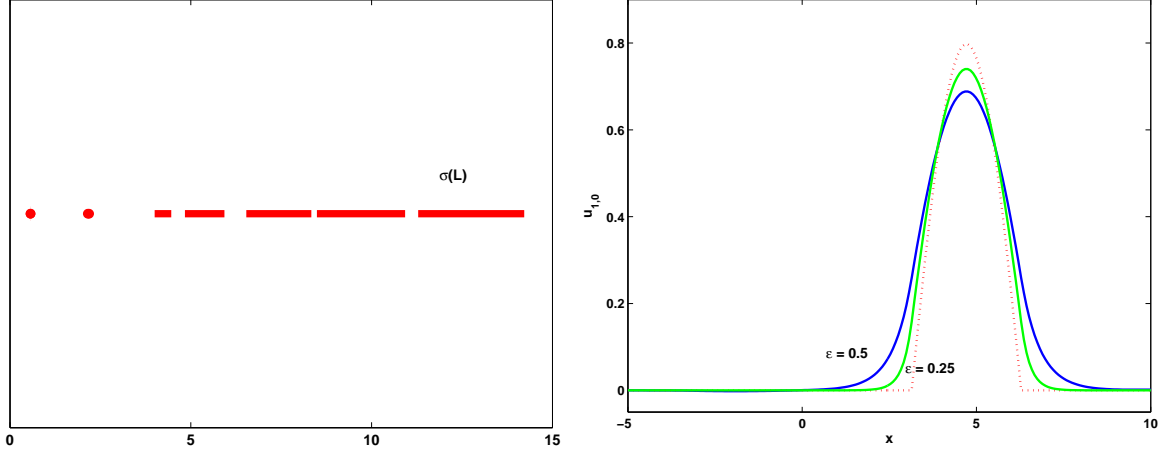


Figure 1: Left: the band-gap structure of the spectrum $\sigma(L)$ for $\varepsilon = 0.5$. Right: the Wannier functions $\hat{u}_{l,0}(x)$ for $\varepsilon = 0.5$ and $\varepsilon = 0.25$. The dotted line shows the asymptotic approximation (2.11).

Let us assume that the sequence $\{\phi_n\}_{n \in \mathbb{Z}}$ satisfies the DNLS equation (1.3) with $\alpha = \frac{\hat{\omega}_{l,1}}{\mu}$ and $\beta = \|\hat{u}_{l,0}\|_{L^4(\mathbb{R})}^4$. Since the Wannier functions satisfy the ODE system from [7]

$$-\hat{u}_{l,n}''(x) + V(x)\hat{u}_{l,n}(x) = \sum_{n' \in \mathbb{Z}} \hat{\omega}_{l,n-n'} \hat{u}_{l,n'}(x), \quad \forall n \in \mathbb{Z}, \quad (2.14)$$

we obtain an inhomogeneous PDE system for the function $\varphi(x, t)$ in the form

$$\begin{aligned} i\varphi_t = & -\varphi_{xx} + V(x)\varphi - \hat{\omega}_{l,0}\varphi + \frac{1}{\mu} \sum_{n \in \mathbb{Z}} \sum_{m \geq 2} \hat{\omega}_{l,m} (\phi_{n+m} + \phi_{n-m}) \hat{u}_{l,n} \\ & + \sigma \left(|\varphi_0 + \mu\varphi|^2 (\varphi_0 + \mu\varphi) - \beta \sum_{n \in \mathbb{Z}} |\phi_n|^2 \phi_n \hat{u}_{l,n} \right). \end{aligned} \quad (2.15)$$

The term $|\varphi_0|^2 \varphi_0$ gives projections both to the selected l -th spectral band and to its complement in $L^2(\mathbb{R})$. The following two lemmas allow us to control both projections.

Lemma 1 *Let E_l be the invariant closed subspace of $L^2(\mathbb{R})$ associated with the l -th spectral band and assume that $E_l \cap E_m = \emptyset$ for a fixed $l \in \mathbb{N}$ and all $m \neq l$. Then, $\langle \hat{u}_{n,l}, \hat{u}_{n',l} \rangle = \delta_{n,n'}$ for any $n, n' \in \mathbb{Z}$ and there exists constants $\eta_l > 0$ and $C_l > 0$, such that*

$$|\hat{u}_{l,n}(x)| \leq C_l e^{-\eta_l |x - 2\pi n|}, \quad \forall n \in \mathbb{Z}, \quad \forall x \in \mathbb{R}. \quad (2.16)$$

Moreover, if $\vec{\phi} \in l^1(\mathbb{Z})$, $\hat{u}_{l,n} \in \mathcal{H}^1(\mathbb{R})$, and $\phi(x) = \sum_{n \in \mathbb{Z}} \phi_n \hat{u}_{l,n}(x)$ for a fixed $l \in \mathbb{N}$, then $\phi \in E_l$, $(\phi, \psi) = 0$, $\forall \psi \in \cup_{m \neq l} E_m$, and $\phi \in \mathcal{H}^1(\mathbb{R})$.

Proof. The orthogonality and exponential decay of Wannier functions follows from the orthogonality relations (2.1) and complex integration (see [7] for the proof). The assertion that $\phi \in E_l$ and $(\phi, \psi) = 0$, $\forall \psi \in \cup_{m \neq l} E_m$ follows from the L^2 spectral theory for the operator $L = -\partial_x^2 + V(x)$ (if $\vec{\phi} \in l^1(\mathbb{Z})$, then $\vec{\phi} \in l^2(\mathbb{Z})$ and $\phi \in L^2(\mathbb{R})$). The assertion that $\phi \in \mathcal{H}^1(\mathbb{R})$ follows from the triangular inequality. \square

Remark 1 According to property (i) of Proposition 1, the l -th spectral band for a fixed $l \in \mathbb{N}$ becomes disjoint from the rest of the spectrum of L in the asymptotic limit $\varepsilon \rightarrow 0$. According to property (i) of Proposition 2, $\hat{u}_{l,n} \in \mathcal{H}^1(\mathbb{R})$ for all $n \in \mathbb{N}$ uniformly in $\varepsilon \geq 0$. Therefore, the assumptions of Lemma 1 are satisfied for sufficiently small $\varepsilon \geq 0$.

Lemma 2 Let Π be an orthogonal projection from $L^2(\mathbb{R})$ to $E_l \subset L^2(\mathbb{R})$. There exists a unique solution $\varphi \in \mathcal{H}^1(\mathbb{R})$ of the inhomogeneous equation

$$(-\partial_x^2 + V(x) - \hat{\omega}_{l,0}) \varphi = (\mathcal{I} - \Pi) f, \quad (2.17)$$

for any $f \in L^2(\mathbb{R})$, uniformly in $\varepsilon \geq 0$, such that $(\varphi, \psi) = 0$, $\forall \psi \in E_l$.

Proof. By property (i) of Proposition 1, if $f \in L^2(\mathbb{R})$, then $\varphi \in L^2(\mathbb{R})$ uniformly in $\varepsilon \geq 0$. By property (ii) of Proposition 1, we obtain

$$\|\varphi'\|_{L^2(\mathbb{R})}^2 + \|V^{1/2}\varphi\|_{L^2(\mathbb{R})}^2 \leq |\hat{\omega}_{l,0}| \|\varphi\|_{L^2(\mathbb{R})}^2 + |(\varphi, f)| \leq C \|f\|_{L^2(\mathbb{R})}^2, \quad (2.18)$$

where the constant $C > 0$ is ε -independent. Therefore, if $f \in L^2(\mathbb{R})$, then $\varphi \in \mathcal{H}^1(\mathbb{R})$ uniformly in $\varepsilon \geq 0$. Uniqueness of φ follows from the fact that the operator $L - \hat{\omega}_{l,0}$ is invertible in $L^2(\mathbb{R}) \setminus E_l$. \square

We can also use the following elementary result.

Lemma 3 The space $\mathcal{H}^1(\mathbb{R})$ forms Banach algebra under the pointwise multiplication, such that

$$\forall u, v \in \mathcal{H}^1(\mathbb{R}) : \quad \|uv\|_{\mathcal{H}^1(\mathbb{R})} \leq C \|u\|_{\mathcal{H}^1(\mathbb{R})} \|v\|_{\mathcal{H}^1(\mathbb{R})}, \quad (2.19)$$

for some $C > 0$.

Proof. The result follows from the representation $\|u\|_{\mathcal{H}^1(\mathbb{R})}^2 = \|u\|_{H^1(\mathbb{R})}^2 + \|V^{1/2}u\|_{L^2(\mathbb{R})}^2$ and the Sobolev embedding theorem $\|u\|_{L^\infty(\mathbb{R})} \leq C \|u\|_{H^1(\mathbb{R})}$ for some $C > 0$. \square

Let us return back to the evolution problem (2.15) and decompose the solution $\varphi(x, t)$ into two parts $\varphi(x, t) = \varphi_1(x, T) + \psi(x, t)$, where φ_1 is a solution of the inhomogeneous equation (2.17) with $f = -\sigma |\varphi_0(x, T)|^2 \varphi_0(x, T)$, while ψ satisfies the evolution problem in the abstract form

$$i\psi_t = (L - \hat{\omega}_{l,0}) \psi + \mu R(\vec{\phi}) + \mu \sigma N(\vec{\phi}, \psi), \quad (2.20)$$

where

$$R(\vec{\phi}) = \frac{1}{\mu^2} \sum_{n \in \mathbb{Z}} \sum_{m \geq 2} \hat{\omega}_{l,m} (\phi_{n+m} + \phi_{n-m}) \hat{u}_{l,n} + \frac{\sigma}{\mu} \left(\Pi |\varphi_0|^2 \varphi_0 - \beta \sum_{n \in \mathbb{Z}} |\phi_n|^2 \phi_n \hat{u}_{l,n} \right) \quad (2.21)$$

and

$$\begin{aligned} N(\vec{\phi}, \psi) = & -i\sigma \partial_T \varphi_1 + 2|\varphi_0|^2 (\varphi_1 + \psi) + \varphi_0^2 (\bar{\varphi}_1 + \bar{\psi}) \\ & + \mu (2|\varphi_1 + \psi|^2 \varphi_0 + (\varphi_1 + \psi)^2 \bar{\varphi}_0) + \mu^2 |\varphi_1 + \psi|^2 (\varphi_1 + \psi), \end{aligned} \quad (2.22)$$

with $\varphi_0 = \sum_{n \in \mathbb{N}} \phi_n \hat{u}_{l,n}$, $\varphi_1 = -\sigma (\mathcal{I} - \Pi)(L - \hat{\omega}_{l,0})^{-1} (\mathcal{I} - \Pi) |\varphi_0|^2 \varphi_0$, and $\sigma = \pm 1$. The following lemma gives a bound on the vector field of the evolution problem (2.20).

Lemma 4 Let $D_{\delta_1} \subset l^1(\mathbb{Z})$ be a ball of finite radius δ_1 centered at $0 \in l^1(\mathbb{Z})$, $D_{\delta_2} \subset \mathcal{H}^1(\mathbb{R})$ be a ball of finite radius δ_2 centered at $0 \in \mathcal{H}^1(\mathbb{R})$ and $R_{\mu_0} \subset \mathbb{R}$ be an interval of small radius μ_0 centered at $0 \in \mathbb{R}$. Then, for any $\mu \in (0, \mu_0)$, $\|\vec{\phi}\|_{l^1(\mathbb{Z})} \in [0, \delta_1)$ and $\|\psi\|_{\mathcal{H}^1(\mathbb{R})} \in [0, \delta_2)$, there exists μ -independent constants $C_R, C_N > 0$ such that

$$\|R(\vec{\phi})\|_{\mathcal{H}^1(\mathbb{R})} \leq C_R \|\vec{\phi}\|_{l^1(\mathbb{Z})}, \quad \|N(\vec{\phi}, \psi)\|_{\mathcal{H}^1(\mathbb{R})} \leq C_N \left(\|\vec{\phi}\|_{l^1(\mathbb{Z})} + \|\psi\|_{\mathcal{H}^1(\mathbb{R})} \right). \quad (2.23)$$

Proof. By the last assertion of Lemma 1, if $\vec{\phi} \in l^1(\mathbb{Z})$ and $\phi(x) = \sum_{n \in \mathbb{Z}} \phi_n \hat{u}_{l,n}(x)$ for a fixed $l \in \mathbb{N}$, then $\phi \in \mathcal{H}^1(\mathbb{R})$. Therefore, there exists $C_0 > 0$ such that $\|\varphi_0\|_{\mathcal{H}^1(\mathbb{R})} \leq C_0 \|\vec{\phi}\|_{l^1(\mathbb{Z})}$. By Lemmas 2 and 3, there exists $C_1 > 0$ such that $\|\varphi_1\|_{\mathcal{H}^1(\mathbb{R})} \leq C_1 \|\vec{\phi}\|_{l^1(\mathbb{Z})}^3$, where we have used the fact that $\|\varphi_0\|_{\mathcal{H}^1(\mathbb{R})}^2 \|\varphi_0\|_{\mathcal{H}^1(\mathbb{R})} \leq C^3 \|\varphi_0\|_{\mathcal{H}^1(\mathbb{R})}^3$ for some $C > 0$. The vector field $R(\vec{\phi})$ can be represented by $R(\vec{\phi}) = \sum_{n \in \mathbb{Z}} r_n(\vec{\phi}) \hat{u}_{l,n}(x)$, where

$$r_n(\vec{\phi}) = \frac{1}{\mu^2} \sum_{m \geq 2} \hat{\omega}_{l,m} (\phi_{n+m} + \phi_{n-m}) + \frac{\sigma}{\mu} \sum_{(n_1, n_2, n_3) \in \mathbb{Z}^3 \setminus \{(n, n, n)\}} K_{n, n_1, n_2, n_3} \phi_{n_1} \bar{\phi}_{n_2} \phi_{n_3},$$

where $K_{n, n_1, n_2, n_3} = (\hat{u}_{l,n}, \hat{u}_{l,n_1} \hat{u}_{l,n_2} \hat{u}_{l,n_3})$. The first bound in (2.23) is proved if $\vec{r} \in l^1(\mathbb{Z})$ for every $\vec{\phi} \in l^1(\mathbb{Z})$ and the map $\vec{r}(\vec{\phi})$ is uniformly bounded for small $\mu > 0$. The first term in $\vec{r}(\vec{\phi})$ is estimated as follows

$$\left\| \sum_{m \geq 2} \hat{\omega}_{l,m} (\phi_{n+m} + \phi_{n-m}) \right\|_{l^1(\mathbb{Z})} \leq \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z} \setminus \{0, 1, -1\}} |\hat{\omega}_{l, m+n}| |\phi_n| \leq K_1 \|\vec{\phi}\|_{l^1(\mathbb{Z})},$$

where $K_1 = \sup_{n \in \mathbb{Z} \setminus \{0, 1, -1\}} \sum_{m \in \mathbb{Z}} |\hat{\omega}_{l, n+m}|$. Since $\omega_l(k)$ is analytically extended along the Riemann surface on $k \in \mathbb{T}$ (by Theorem XIII.95 on p.301 in [8]), we have $\omega_l \in H^s(\mathbb{T})$ for any $s \geq 0$, such that $K_1 < \infty$. The second term in $\vec{r}(\vec{\phi})$ is estimated as follows

$$\begin{aligned} \left\| \sum_{(n_1, n_2, n_3) \in \mathbb{Z}^3 \setminus \{(n, n, n)\}} K_{n, n_1, n_2, n_3} \phi_{n_1} \bar{\phi}_{n_2} \phi_{n_3} \right\|_{l^1(\mathbb{Z})} &\leq \sum_{n \in \mathbb{Z}} \sum_{(n_1, n_2, n_3) \in \mathbb{Z}^3 \setminus \{(n, n, n)\}} |K_{n, n_1, n_2, n_3}| |\phi_{n_1}| |\phi_{n_2}| |\phi_{n_3}| \\ &\leq K_2 \|\vec{\phi}\|_{l^1(\mathbb{Z})}^3, \end{aligned}$$

where $K_2 = \sup_{(n_1, n_2, n_3) \in \mathbb{Z}^3 \setminus \{(n, n, n)\}} \sum_{n \in \mathbb{Z}} |K_{n, n_1, n_2, n_3}|$. Using the exponential decay (2.16), we obtain

$$\sum_{n \in \mathbb{Z}} |\hat{u}_{l,n}(x)| \leq C_l \sum_{n \in \mathbb{Z}} e^{-\eta_l |x - 2\pi n|} \leq A_l$$

for some $A_l > 0$ uniformly in $x \in \mathbb{R}$ and

$$\sum_{n \in \mathbb{Z}} |K_{n, n_1, n_2, n_3}| \leq A_l \int_{\mathbb{R}} |\hat{u}_{l,n_1}(x)| |\hat{u}_{l,n_2}(x)| |\hat{u}_{l,n_3}(x)| dx \leq A_l \|\hat{u}_{l,0}\|_{L^4(\mathbb{R})}^2 \|\hat{u}_{l,0}\|_{L^2(\mathbb{R})}$$

uniformly in $(n_1, n_2, n_3) \in \mathbb{Z}^3$. By the Sobolev embedding theorem and property (i) of Proposition 2, $\|\hat{u}_{l,0}\|_{L^4(\mathbb{R})} \leq C \|\hat{u}_{l,0}\|_{H^1(\mathbb{R})} \leq C \|\hat{u}_{l,0}\|_{\mathcal{H}^1(\mathbb{R})}$ for some $C > 0$, such that $K_2 < \infty$. Therefore, the norm $\|R(\vec{\phi})\|_{\mathcal{H}^1(\mathbb{R})}$ is bounded from above by the norm $\|\vec{\phi}\|_{l^1(\mathbb{Z})}$.

To show that the constant C_R is uniform for small $\mu > 0$, we use Propositions 1 and 2. By property (iii) of Proposition 1, $\hat{\omega}_{l,m} = O(\mu^m)$ for all $m \geq 2$, such that K_1/μ^2 is uniformly bounded for small μ .

By property (iii) of Proposition 2, $K_{n,n_1,n_2,n_3} = O(\mu^{|n_1-n|+|n_2-n|+|n_3-n|+|n_2-n_1|+|n_3-n_1|+|n_3-n_2|})$ for all $n_1, n_2, n_3 \in \mathbb{Z}^3 \setminus \{(n, n, n)\}$, such that K_2/μ is uniformly bounded for small μ . Thus, the first bound in (2.23) is proved.

The second bound in (2.23) follows from the fact that both $\mathcal{H}^1(\mathbb{R})$ and $l^1(\mathbb{Z})$ form Banach algebras with respect to pointwise multiplication. As a result, if $\vec{\phi} \in l^1(\mathbb{Z})$ and $\vec{\phi}(T)$ is a solution of the DNLS equation (1.3), then $\partial_T \vec{\phi} \in l^1(\mathbb{Z})$ and if $\varphi_0, \varphi_1 \in \mathcal{H}^1(\mathbb{R})$ and $\vec{\phi} \in l^1(\mathbb{Z})$, then $N(\vec{\phi}, \psi)$ maps $\psi \in \mathcal{H}^1(\mathbb{R})$ to an element of $\mathcal{H}^1(\mathbb{R})$. \square

We can now prove that the initial-value problem for the time-evolution equation (2.20) and the initial-value problem for the DNLS equation (1.3) are locally well-posed.

Theorem 1 *Let $\vec{\phi}(T) \in C^1(\mathbb{R}, l^1(\mathbb{Z}))$ and $\psi_0 \in \mathcal{H}^1(\mathbb{R})$. Then, there exists a $t_0 > 0$ and a unique solution $\psi(t) \in C^1([0, t_0], \mathcal{H}^1(\mathbb{R}))$ of the time-evolution problem (2.20) with initial data $\psi(x, 0) = \psi_0(x)$.*

Proof. Since L is a self-adjoint operator, the operator $e^{-it(L-\hat{\omega}_{l,0})}$ forms a strongly continuous semi-group and

$$\|e^{-it(L-\hat{\omega}_{l,0})}\| \leq K_0,$$

for some $K_0 > 0$ uniformly in $t \in \mathbb{R}_+$. Using the variation of constant formula, we rewrite the time-evolution problem (2.20) in the integral form

$$\psi(t) = e^{-it(L-\hat{\omega}_{l,0})}\psi_0 + \int_0^t e^{-i(t-s)(L-\hat{\omega}_{l,0})} \left(\mu R(\vec{\phi}(s)) + \mu \sigma N(\vec{\phi}(s), \psi(s)) \right) ds. \quad (2.24)$$

By using bounds (2.23) on $R(\vec{\phi})$ and $N(\vec{\phi}, \psi)$ and the contraction mapping principle for sufficiently small $t_0 > 0$, one can show with a standard analysis that there exists a unique solution $\psi(t) \in C^1([0, t_0], \mathcal{H}^1(\mathbb{R}))$ of the integral equation (2.24). \square

Theorem 2 *Let $\vec{\phi}_0 \in l^1(\mathbb{Z})$. Then, there exist a $T_0 > 0$ and a unique solution $\vec{\phi}(T) \in C^1([0, T_0], l^1(\mathbb{Z}))$ of the DNLS equation (1.3) with initial data $\vec{\phi}(0) = \vec{\phi}_0$.*

Proof. By the variation of constant formula, we have

$$\vec{\phi}(T) = \vec{\phi}_0 - i \int_0^T \left(\alpha \Delta \vec{\phi}(s) + \sigma \beta \Gamma(\vec{\phi}(s)) \right) ds,$$

where $(\Delta \vec{\phi})_n = \phi_{n+1} + \phi_{n-1}$ and $(\Gamma(\vec{\phi}))_n = |\phi_n|^2 \phi_n$. Since $l^1(\mathbb{Z})$ forms a Banach algebra, the right-hand-side of the integral equation maps an element of $l^1(\mathbb{Z})$ to an element of $l^1(\mathbb{Z})$. Therefore, there exists a unique solution $\vec{\phi}(T) \in C^1([0, T_0], l^1(\mathbb{Z}))$ of the integral equation for sufficiently small $T_0 > 0$. \square

We can now formulate the main theorem of our article.

Theorem 3 *Fix $l \in \mathbb{N}$ and let $\vec{\phi}(T) \in C^1([0, T_0], l^1(\mathbb{Z}))$ be a solution of the DNLS equation (1.3) with initial data $\vec{\phi}(0) = \vec{\phi}_0$ satisfying the bound*

$$\left\| \phi_0 - \mu^{1/2} \sum_{n \in \mathbb{Z}} \phi_n(0) \hat{u}_{l,n}(x) \right\|_{\mathcal{H}^1(\mathbb{R})} \leq C_0 \mu^{3/2} \quad (2.25)$$

for some $C_0 > 0$. Then, for any $\mu \in (0, \mu_0)$ with sufficiently small $\mu_0 > 0$, there exists a μ -independent constant $C > 0$ such that equation (1.1) has a solution $\phi(t) \in C^1([0, T_0/\mu], \mathcal{H}^1(\mathbb{R}))$ satisfying the bound

$$\forall t \in [0, T_0/\mu] : \left\| \phi(\cdot, t) - \mu^{1/2} \sum_{n \in \mathbb{Z}} \phi_n(T) \hat{u}_{l,n} \right\|_{\mathcal{H}^1(\mathbb{R})} \leq C \mu^{3/2}. \quad (2.26)$$

Remark 2 Since $\mu = \varepsilon e^{-a/\varepsilon}$, the finite interval $[0, T_0/\mu]$ is exponentially large with respect to parameter ε , similar to the bounds obtained in [2].

Theorem 3 is proved in the following section.

3 Bounds on the remainder terms

We develop the proof of Theorem 3 by using the energy estimates for the time-evolution problem (2.20) and Gronwall's inequality for a scalar first-order differential equation. The GP equation (1.1) has two conserved quantities

$$Q(\phi) = \int_{\mathbb{R}} |\phi|^2 dx, \quad E(\phi) = \int_{\mathbb{R}} \left(|\phi_x|^2 + V(x) |\phi|^2 + \frac{1}{2} \sigma |\phi|^4 \right) dx, \quad (3.1)$$

which have the meaning of the charge and energy invariants, such that $Q(\phi) = Q(\phi_0)$ and $E(\phi) = E(\phi_0)$ for any solution $\phi(x, t)$ starting from the initial data $\phi_0(x)$. We shall consider the quantity $E_Q(\psi) = \|\psi\|_{\mathcal{H}^1(\mathbb{R})}^2$, which is not a constant in t if $\psi(x, t)$ satisfies the time-evolution problem (2.20). The time evolution of $E_Q(\psi)$ obeys the following estimate.

Lemma 5 Let $\vec{\phi}(T) \in C^1(\mathbb{R}_+, l^1(\mathbb{Z}))$ be any function and $\psi(t) \in C^1([0, t_0], \mathcal{H}^1(\mathbb{R}))$ be a local solution of the time-evolution problem (2.20) for some $t_0 > 0$. Then, for any $\mu \in [0, 1]$ and every $M > 0$, there exist a μ -independent constant $C_E > 0$ such that

$$\left| \frac{d}{dt} E_Q(\psi) \right| \leq \mu C_E \left(\|\vec{\phi}\|_{l^1(\mathbb{Z})} + E_Q(\psi) \right) \quad (3.2)$$

as long as $E_Q(\psi) \leq M$.

Proof. By direct differentiation, for any $\psi(t) \in C^1([0, t_0], \mathcal{H}^1(\mathbb{R}))$, we have

$$\begin{aligned} \frac{d}{dt} \|\psi\|_{\mathcal{H}^1(\mathbb{R})}^2 &= -i\mu \int_{\mathbb{R}} \left(\bar{\psi}_x R_x(\vec{\phi}) - \psi_x \bar{R}_x(\vec{\phi}) \right) dx - i\mu\sigma \int_{\mathbb{R}} \left(\bar{\psi}_x N_x(\vec{\phi}, \psi) - \psi_x \bar{N}_x(\vec{\phi}, \psi) \right) dx \\ &\quad - i\mu \int_{\mathbb{R}} (1 + V(x)) \left(\bar{\psi} R(\vec{\phi}) - \psi \bar{R}(\vec{\phi}) \right) dx \\ &\quad - i\mu\sigma \int_{\mathbb{R}} (1 + V(x)) \left(\bar{\psi} N(\vec{\phi}, \psi) - \psi \bar{N}(\vec{\phi}, \psi) \right) dx. \end{aligned}$$

Using the Cauchy–Schwartz inequalities and the bounds (2.23) of Lemma 4, we obtain

$$\left| \frac{d}{dt} \|\psi\|_{\mathcal{H}^1(\mathbb{R})}^2 \right| \leq 6\mu \|\psi\|_{\mathcal{H}^1(\mathbb{R})} \left((C_R + C_N) \|\vec{\phi}\|_{l^1(\mathbb{Z})} + C_N \|\psi\|_{\mathcal{H}^1(\mathbb{R})} \right),$$

where $\sigma = \pm 1$ has been used. By canceling one power of $\|\psi\|_{\mathcal{H}^1(\mathbb{R})}$, we arrive to the bound (3.2). \square

Theorem 3 is then a direct consequence of the following corollary.

Corollary 1 *A local solution $\psi(t) \in C^1([0, T_0/\mu], \mathcal{H}^1(\mathbb{R}))$ of the time-evolution problem (2.20) for any $\psi(0) \in \mathcal{H}^1(\mathbb{R})$ and any $\vec{\phi}(T) \in C^1([0, T_0], l^1(\mathbb{Z}))$ satisfies the bound*

$$\sup_{t \in [0, T_0/\mu]} \|\psi(t)\|_{\mathcal{H}^1(\mathbb{R})} \leq \left(\|\psi(0)\|_{\mathcal{H}^1(\mathbb{R})} + C_E T_0 \sup_{T \in [0, T_0]} \|\vec{\phi}(T)\|_{l^1(\mathbb{Z})} \right) e^{C_E T_0}. \quad (3.3)$$

Proof. By using the bound (3.2), we obtain

$$E_Q(\psi(t)) \leq E_Q(\psi_0) + \mu C_E \int_0^t \left(\|\vec{\phi}(\mu s)\|_{l^1(\mathbb{Z})} + E_Q(\psi(s)) \right) ds.$$

The bound (3.3) follows by Gronwall's inequality on $t \in [0, T_0/\mu]$. \square

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